ON OSSA'S THEOREM AND LOCAL PICARD GROUPS

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Abstract. We analyze the stable isomorphism type of polynomial rings as modules over the subalgebra $\mathcal{A}(1) = \langle Sq^1, Sq^2 \rangle$ of the Steenrod algebra. This provides the real case of Erich Ossa's theorem, which gives a calculation of the real connective K-theory of elementary abelian 2-groups. We also study the Picard groups of the Q_i-local subcategories, generalizing results of Cherng-Yih Yu, and determine the idempotents in the category of bounded below $\mathcal{A}(1)$-modules of finite type. The results here are used in Geoffrey Powell's analysis of connective real K-theory considered as a polynomial functor on the category of $\mathbf{F}_2$-vector spaces.

Contents

1. Introduction

In 1987, Erich Ossa ([8]) showed that $P = H^*BC_2$ is stably idempotent as a module over the subalgebra $E(1) = E[Q_0, Q_1]$ of the Steenrod algebra. He used this to show that if V is an elementary abelian group then, modulo Bott torsion, the connective complex K-theory of BV_+ is the completion of the Rees ring of the representation ring $R(V)$ with respect to its augmentation ideal. (This is not how he said it, and his main focus was on related topological results, but this is one way of phrasing the first theorem in [8].) He tried to extend this to real connective K-theory, but there were flaws in his argument. By 1992, Stephan Stolz (private communication) knew that the correct statement for the real case was that $P^{\otimes(n+1)}$ was the n^{th} syzygy of P in the category of $\mathcal{A}(1)$-modules. These syzygies and their periodicity were explicitly identified in Margolis ([7, Chap. 23]), but had already been visible as early as the 1968 paper [6] by Gitler, Mahowald and Milgram, though the periodicity was not stated there. The relation to the tensor powers of P was not evident in this earlier work. In his unpublished 1995 Notre Dame PhD thesis, Cherng-Yih Yu ([12])) gave a proof of these facts together with the remarkable fact that these $\mathcal{A}(1)$-modules form the Picard group of the category of finitely generated, bounded below, Q_1-local $\mathcal{A}(1)$-modules: $\mathrm{Pic}^{(1)}(\mathcal{A}(1)) = \{\Sigma^i P^{\otimes(n)}\} \cong \mathbf{Z} \oplus \mathbf{Z}/(4)$ in our notation (Definition 8.1). As with Ossa's result in the complex case, this should lead to a representation theoretic description of the real connective K-theory of BV_+ modulo Bott torsion. However, this was found by other means in the author's joint work with John Greenlees ([4, p. 177]). More recently, Geoffrey Powell has given descriptions of the real and complex connective K-homology and cohomology of BV_+ as polynomial

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functors in [9] and [10]. His functorial approach provides significant simplifications. Some of the results here are used in his work on the real case.

In the present work, we start from the beginning and give a complete account of the structure of the tensor powers $P^{\otimes(n)}$ as $\mathcal{A}(1)$-modules. This is equivalent to an analysis of the $\mathcal{A}(1)$-module structure of the polynomial rings $\mathbf{F}_2[x_1, \ldots, x_n]$, and this may find application in the study of the hit problem. For example, we identify the polynomial in $\mathbf{F}_2[x_1, x_2, x_3, x_4]$ responsible for Bott periodicity in the real connective K-theory of elementary abelian groups (Theorem 6.1). We believe that our account has the merit of latter day presentations in its clarity and simplicity. We take Yu's idea of studying Picard groups of local subcategories of $E(1)$-modules and $\mathcal{A}(1)$-modules seriously. We study Q_0 and Q_1 local $\mathcal{A}(1)$-modules and the Picard groups $\mathrm{Pic}^{(i)}(B)$ for $i = 0$ or 1 and $B = E(1)$ or $\mathcal{A}(1)$. We should point out that the hard work of identifying the elements of $\mathrm{Pic}^{(1)}(\mathcal{A}(1))$ is quoted from Yu ([12]). We have no substantial simplification of this to offer. We show that the Picard group of the whole category maps injectively into the sum of these local Picard groups. Finally, we show that we have found all stably idempotent modules which could give rise to such localizations (Theorem 10.1). In the process, we give more general results about the structure of $\mathcal{A}(1)$-modules which we hope will be of independent interest. Among these are Theorems 3.2 and 7.4, Corollary 7.6, the explicit $\mathcal{A}(1)$ (and hence $\mathcal{A}$) resolution of P in Figure 3, and the results of Sections 8 to 10.

This work has evolved fitfully over the years since [5], to which it provides context and additional detail, receiving one impetus from my joint work with John Greenlees ([3] and [4]), another from questions asked by Vic Snaith (which led to [11]), and a more recent one from discussions with Geoffrey Powell in connection with [10]. I am grateful to Geoffrey Powell for many useful discussions while working out the final versions of these results and to the University of Paris 13 for the opportunity to do this in May of 2012.

Stephan Stolz was responsible for getting me started thinking about these matters and has certainly known many of the results here for many years.

2. Basic definitions and results

We begin with some basic definitions and results about modules over finite sub-Hopf algebras of the Steenrod algebra, in order to state clearly the hypotheses under which they hold. The reader who is familiar with $\mathcal{A}(1)$-modules should probably skip to the next section.

Let $\mathcal{A}(n)$ be the subalgebra of the mod 2 Steenrod algebra $\mathcal{A}$ generated by $\{Sq^{2^i} \mid 0 \le i \le n\}$. Thus $\mathcal{A}(0)$ is exterior on one generator, Sq^1, and $\mathcal{A}(1)$, generated by Sq^1 and Sq^2, is 8 dimensional.

Let $E(n)$ be the exterior subalgebra of $\mathcal{A}$ generated by the Milnor primitives $\{Q_i \mid 0 \le i \le n\}$. (Recall that $Q_0 = Sq^1$ and $Q_n = Sq^{2^n} Q_{n-1} + Q_{n-1} Sq^{2^n}$.) $E(n)$ is a sub-Hopf algebra of $\mathcal{A}(n)$.

By *module* we will mean left module, and by *ideal*, left ideal.

For $B = E(1)$, $\mathcal{A}(1)$, or any finite sub-Hopf algebra of $\mathcal{A}$, let B-Mod be the category of all graded B-modules. The category B-Mod is abelian, complete, cocomplete, and has enough projectives and injectives. Since B is a Frobenius algebra, free, projective and injective are equivalent conditions in B-Mod. (See Margolis ([7]), Chapters 12, 13 and 15, and in particular Lemma 15.27 for details.)

The best results hold in the abelian subcategory $B\text{-Mod}^f$ of bounded-below B-modules of finite type. It has enough projectives and injectives ([7, Lemma 15.27]). A module in $B\text{-Mod}^f$ is free, projective, or injective there iff it is so in B-Mod ([7, Lemma 15.17]). The Krull-Schmidt property holds in $B\text{-Mod}^f$.

Proposition 2.1. *(*[7, Theorem 11.21]*) A module in $B\text{-Mod}^f$ can be written as a direct sum of indecomposable modules, uniquely up to order and isomorphism.*

Since the algebras B we are considering are Poincare duality algebras, the following decomposition result holds without restriction on M. It will be useful in our discussion of stable isomorphism.

Proposition 2.2. *(*[7, Proposition 13.13 and p. 203]*) A module M in B-Mod has an expression*

$$M \;\cong\; F \oplus M^{\mathrm{red}},$$

unique up to isomorphism, where F is free and M^{red} has no free summands.

Definition 2.3. *We call M^{red} the* reduced *part of M.*

Note that we are not asserting that $M \mapsto M^{\mathrm{red}}$ is a functor, or that there are *natural* maps $M \longrightarrow M^{\mathrm{red}}$ or $M^{\mathrm{red}} \longrightarrow M$.

Definition 2.4. *If $\mathcal{C}$ is a subcategory of B-*Mod*, the* stable module category of $\mathcal{C}$*, written* $\mathrm{St}(\mathcal{C})$*, is the category with the same objects as $\mathcal{C}$ and with morphisms replaced by their equivalence classes modulo those which factor through a projective module. Let us write $M \simeq N$ to denote* stable isomorphism*, isomorphism in* $\mathrm{St}(B\text{-Mod})$*, and reserve $M \cong N$ for isomorphism in B-*Mod.

Over a finite Hopf algebra like B, stable isomorphism simplifies.

Proposition 2.5. *(*[7, Proposition 14.1]*) In B-*Mod*, modules M and N are stably isomorphic iff there exist free modules P and Q such that $M \oplus P \cong N \oplus Q$.*

In $B\text{-Mod}^f$, stable isomorphism simplifies further.

Proposition 2.6. *(*[7, Proposition 14.11] *Let M and N be modules in $B\text{-Mod}^f$.*

(1) *$M \simeq N$ iff $M^{\mathrm{red}} \cong N^{\mathrm{red}}$.*
(2) *$f : M \longrightarrow N$ is a stable equivalence iff $M^{\mathrm{red}} \rightarrowtail M \xrightarrow{f} N \twoheadrightarrow N^{\mathrm{red}}$ is an isomorphism in B-*Mod.

Here, $M^{\mathrm{red}} \rightarrowtail M$ and $N \twoheadrightarrow N^{\mathrm{red}}$ are any maps which are part of a splitting of M and N, respectively, into a free summand and a reduced summand.

The preceding result holds for all finite Hopf algebras. For modules over subalgebras B of the Steenrod algebra, the theorem of Adams and Margolis ([1] or [7, Theorem 19.6]) gives us a simple criterion for stable isomorphism in $B\text{-Mod}^f$. Recall that the Milnor primitives Q_i satisfy $Q_i^2 = 0$, so that we may define $H(M, Q_i) = \mathrm{Ker}(Q_i)/\mathrm{Im}(Q_i)$.

Theorem 2.7. *Let $B = \mathcal{A}(1)$ or $E(1)$. Suppose that $f : M \longrightarrow N$ in $B\text{-Mod}^f$. If f induces isomorphisms $f_* : H(M, Q_i) \longrightarrow H(N, Q_i)$ for $i = 0$ and $i = 1$, then f is a stable isomorphism.*

In particular, if a bounded-below module M has trivial Q_0 and Q_1 homology, then the map $0 \longrightarrow M$ is a stable equivalence, and therefore M is free.

Remark 2.8. *The hypothesis that the modules be bounded-below is needed for Theorem 2.7 to hold: the Laurent series ring $\mathbf{F}_2[x, x^{-1}]$ is not free over $E(1)$ or $\mathcal{A}(1)$, yet has trivial Q_0 and Q_1 homology.*

Margolis ([7, Theorem 19.6.(b)]) gives a similar characterization of stable isomorphism for modules over any sub-Hopf algebra B of the Steenrod algebra.

Finally, we consider the algebraic loops functor. By Schanuel's Lemma, letting ΩM be the kernel of an epimorphism from a projective module to M gives a well defined module up to stable isomorphism. To get functoriality, the following definition is simplest.

Definition 2.9. *Let $I = \mathrm{Ker}(B \longrightarrow \mathbf{F}_2)$ be the augmentation ideal of B. Let $\Omega M = I \otimes M$.*

Similarly, we may define the inverse loops functor.

Definition 2.10. *Let $I^{-1} = \mathrm{Cok}(F_2 \longrightarrow \Sigma^{-d}B))$ be the cokernel of the inclusion of the socle into B. (d is 4 if $B = E(1)$, 6 if $B = \mathcal{A}(1)$.) Let $\Omega^{-1}M = I^{-1} \otimes M$.*

To see that the notation makes sense, recall the 'untwisting' isomorphism

$$\theta : B \otimes M \longrightarrow B \otimes \widehat{M},$$

given by $\theta(b \otimes m) = \sum b' \otimes b''m$. Here $B \otimes \widehat{M}$ is the free B-module on the underlying vector space $\widehat{M}$ of M and $\psi(b) = \sum b' \otimes b''$ is the coproduct of b. The inverse, $\theta^{-1}(b \otimes m) = \sum b' \otimes \chi(b'')m$, where χ is the conjugation (antipode) of B. This shows that tensoring with a free module gives a free module.

In particular, tensor product is well defined in the stable module category.

Tensoring the short exact sequence $0 \longrightarrow I \longrightarrow B \longrightarrow \mathbf{F}_2 \longrightarrow 0$ with I^{-1} shows that $I \otimes I^{-1}$ is stably equivalent to $\mathbf{F}_2$.

Corollary 2.11. *We have stable equivalences $\Omega\Omega^{-1} \simeq \mathrm{Id} \simeq \Omega^{-1}\Omega$. In general, $\Omega^k\Omega^l \simeq \Omega^{k+l}$ for all integers k and l.*

Finally, we should note that the stable module category is triangulated. For any short exact sequence of modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

there is an extension cocycle $\Omega M_3 \longrightarrow M_1$ (or equivalently $M_3 \longrightarrow \Omega^{-1}M_1$). The triangles in the stable module category are the sequences

$$\Omega M_3 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3$$

and

$$M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow \Omega^{-1}M_1.$$

3. Reduction from $P^{\otimes(n)}$ to $\Omega^n P$

Let $P = H^*BC_2 = (x)$, the ideal generated by x in $H^*BC_2+ = \mathbf{F}_2[x]$. Our first step is to show that the tensor powers of P are stably equivalent to the iterated algebraic loops of P. This follows from general results about Q_0-acyclic $A(1)$-modules.

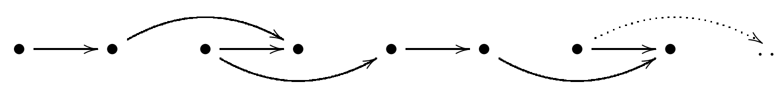
Let R be the subquotient of the Laurent series ring $\mathbf{F}_2[x, x^{-1}]$ which is nonzero in degree -1 and degrees greater than 0. (This is the cohomology of the fiber of the reduced transfer $BC_2 \longrightarrow S$, but we will not need this fact.)

We represent P and R diagrammatically by showing the action of Sq^1 and Sq^2:

$R :$

and

$P :$



The short exact sequence

$$0 \longrightarrow \Sigma^{-1}\mathcal{A}(1)/\!\!/\mathcal{A}(0) \longrightarrow R \longrightarrow \Sigma^4 R \longrightarrow 0$$

gives rise to a filtration of R with associated graded

$$\bigoplus_{i\geq 0} \Sigma^{4i-1}\mathcal{A}(1)/\!\!/\mathcal{A}(0).$$

Here $\mathcal{A}(1)/\!\!/\mathcal{A}(0) = \mathcal{A}(1) \otimes_{\mathcal{A}(0)} \mathbf{F}_2 = \mathcal{A}(1)/(Sq^1)$. Similarly, the filtration $P \supset (x^3) \supset (x^5) \supset \cdots$ has associated graded

$$\bigoplus_{j\geq 0} \Sigma^{2j+1}\mathcal{A}(0).$$

where the isomorphism $\mathcal{A}(0) \cong \mathcal{A}(1)/(Sq^2, Sq^2Sq^1)$ makes $\mathcal{A}(0)$ an $\mathcal{A}(1)$-module. Together, these filtrations yield a useful consequence.

Lemma 3.1.

$$R \otimes P = \bigoplus_{i,j\geq 0} \Sigma^{4i+2j}\mathcal{A}(1)$$

Proof. Since $H(P, Q_0) = 0$ and $H(R, Q_1) = 0$, the Künneth isomorphism implies that $R \otimes P$ has trivial homology with respect to both Q_0 and Q_1, and is therefore free by Theorem 2.6. The precise decomposition into free modules then follows from a Hilbert series calculation, or from the filtrations above, using the untwisting isomorphism $\mathcal{A}(1)/\!\!/\mathcal{A}(0) \otimes \mathcal{A}(0) \cong \mathcal{A}(1)$. □

This lemma combines neatly with the unique non-split short exact sequence

$$0 \longrightarrow P \longrightarrow R \longrightarrow \Sigma^{-1}\mathbf{F}_2 \longrightarrow 0. \tag{1}$$

to give the first step in our argument.

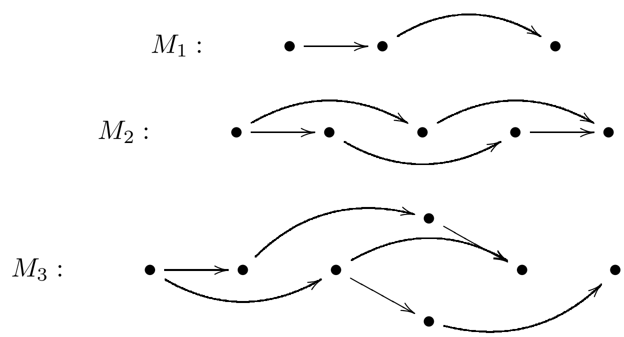


FIGURE 1. The modules M_1, M_2, and M_3 (Def. 4.2)

Theorem 3.2. *If M is Q_0-acyclic, then $M \otimes R$ is free and $M \otimes P$ is stably equivalent to $\Omega\Sigma^{-1}M$. If M is Q_1-acyclic, then $M \otimes P$ is free and $M \otimes R$ is stably equivalent to $\Sigma^{-1}M$.*

Proof. Tensoring our short exact sequence (1) with M we get a short exact sequence

$$0 \longrightarrow M \otimes P \longrightarrow M \otimes R \longrightarrow \Sigma^{-1}M \longrightarrow 0. \tag{2}$$

If M is Q_0-acyclic, then the Künneth theorem implies that both $H(M \otimes R, Q_0)$ and $H(M \otimes R, Q_1)$ are trivial. By Theorem 2.7, $M \otimes R$ is free. It follows that $M \otimes P$ is stably equivalent to $\Omega\Sigma^{-1}M$.

If M is Q_1 acyclic, the same argument shows that $M \otimes P$ is free and hence injective. The sequence therefore splits, and $M \otimes R$ is stably equivalent to $\Sigma^{-1}M$. □

Corollary 3.3. *For $n \geq 0$, the $n+1$-fold tensor product $P^{\otimes(n+1)}$ is stably equivalent to $\Omega^n\Sigma^{-n}P$, and the $n+1$-fold tensor product $R^{\otimes(n+1)}$ is stably equivalent to $\Sigma^{-n}R$.*

Proof. Applying the preceding theorem to $P^{\otimes(n)}$ gives a stable equivalence between $P^{\otimes(n+1)}$ and $\Omega\Sigma^{-1}P^{\otimes(n)}$. The first statement then follows by induction. Applying it to $R^{\otimes(n)}$, we see that $R^{\otimes(n+1)}$ is stably equivalent to $\Sigma^{-1}R^{\otimes(n)}$. The second statement then follows by induction. □

Remark 3.4. *In [8], Lemma 2 asserts that $P \otimes P$ is stably equivalent to $\Sigma^2 P$ rather than $\Sigma^{-1}\Omega P$. This is false in the category of $\mathcal{A}(1)$-modules, though it is true in the category of $E(1)$-modules. These modules differ by one copy of $E(1)$. See Theorems 4.3 and 5.4 here for the correct statements. This also makes Proposition 2 there false, both in identifying the degrees of the Eilenberg-MacLane summands, and in identifying the complement to them.*

4. THE STABLE TYPE OF $P^{\otimes(n)}$

It is now a simple matter to identify the stable type of $P^{\otimes(n)}$ by computing a minimal projective resolution of P. We incorporate the shift appearing in Corollary 3.3 as well.

Definition 4.1. *For $n \in \mathbf{Z}$, let $P_{n+1} = (\Omega^n\Sigma^{-n}P)^{\mathrm{red}}$.*

By Definitions 2.9 and 2.4 and Proposition 2.5, $(\Omega^n M)^{\mathrm{red}}$ is the n^{th} syzygy in a minimal projective resolution of M if $n > 0$ and $M \in \mathcal{A}(1)\text{-Mod}^f$. In particular, Corollary 3.3 says that if $n > 0$, P_n is the indecomposable module in the stable equivalence class of $P^{\otimes(n)}$. It turns out that each of these is an extension of a suspension of R by a finite module, which we define now.

Definition 4.2.

- $M_1 = \Sigma\mathcal{A}(1)/(Sq^2)$
- $M_2 = \Sigma^2\mathcal{A}(1)/(Sq^3)$
- $M_3 = \Sigma^3\mathcal{A}(1)/(Sq^2Sq^2Sq^2)$
- $M_4 = \Sigma^8\mathbf{F}_2$

See Figure 1 for diagrammatic representations of the M_i and Figure 2 for the P_i.

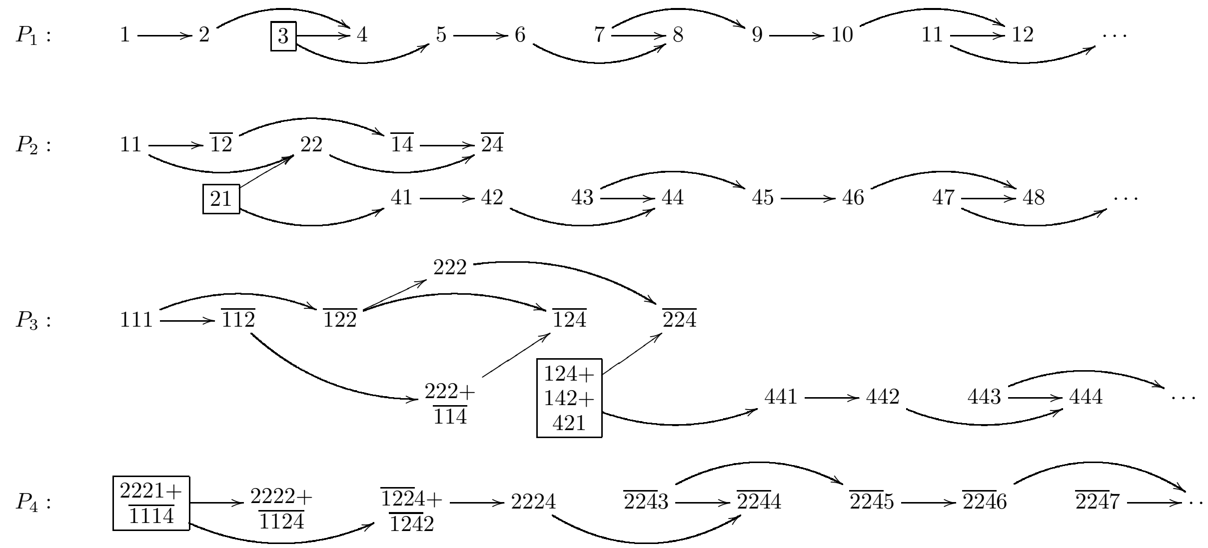


FIGURE 2. The modules P_n embedded in $P^{\otimes(n)}$, $1 \le n \le 4$. The bottom class of the quotient $\Sigma^t R$ is boxed. See Section 6 for notation.

Theorem 4.3. *There are periodicity isomorphisms $\Omega^4 P \simeq \Sigma^{12} P$ and, hence, $P_{n+4} \simeq \Sigma^8 P_n$. There are stable isomorphisms $P_n \otimes P_m \simeq P_{n+m}$. There are short exact sequences*

$$0 \longrightarrow M_1 \longrightarrow P_1 \longrightarrow \Sigma^4 R \longrightarrow 0$$

$$0 \longrightarrow M_2 \longrightarrow P_2 \longrightarrow \Sigma^4 R \longrightarrow 0$$

$$0 \longrightarrow M_3 \longrightarrow P_3 \longrightarrow \Sigma^8 R \longrightarrow 0$$

$$0 \longrightarrow M_4 \longrightarrow P_4 \longrightarrow \Sigma^8 R \longrightarrow 0$$

Each of these is the unique non-trivial extension, with Sq^1 of the bottom class in the suspension of R equal to the unique element of M_i of the relevant degree.

Remark 4.4. *The stable type of $\Omega^n P$, and thus of $P^{\otimes(n)}$, therefore cycles with period 4 (up to suspensions). The simplicity of the complex case stems from the fact that over $E(1)$, the stable type of ΩP is just $\Sigma^3 P$ so that $P \otimes P$ is stably $\Sigma^2 P$, and, more generally, $P^{\otimes(n+1)}$ is stably $\Sigma^{2n} P$.*

Proof. Since P_n is stably equivalent to $P^{\otimes(n)}$ it is evident that $P_n \otimes P_m$ is stably equivalent to P_{n+m}. Now note that tensoring R (or any Q_1-acyclic module) with $\mathcal{A}(0)$ gives a free $\mathcal{A}(1)$-module. Precisely:

$$F = R \otimes \mathcal{A}(0) \;\cong\; \bigoplus_{i \ge 0} \Sigma^{4i-1} \mathcal{A}(1).$$

Then, tensoring the short exact sequence

$$0 \longrightarrow \Sigma \mathbf{F}_2 \longrightarrow \mathcal{A}(0) \longrightarrow \mathbf{F}_2 \longrightarrow 0$$

with R yields a short exact sequence

$$0 \longrightarrow \Sigma R \longrightarrow F \longrightarrow R \longrightarrow 0,$$

which is the start of a minimal resolution of R. (This shows $\Omega R = \Sigma R$.)

The submodule of P generated by the bottom class is M_1 and the quotient by it is $\Sigma^4 R$. This gives

$$0 \longrightarrow M_1 \longrightarrow P \longrightarrow \Sigma^4 R \longrightarrow 0, \tag{3}$$

the first of our claimed short exact sequences. Taking minimal free modules mapping onto the three modules in (3) and applying the snake lemma produces the suspension of the next of our claimed sequences.

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r] & P_1 \arrow[r] & \Sigma^4 R \arrow[r] & 0 \\
0 \arrow[r] & \Sigma\mathcal{A}(1) \arrow[r] \arrow[u] & F_1 \arrow[r] \arrow[u] & \Sigma^4 F \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Sigma M_2 \arrow[r] \arrow[u] & \Sigma P_2 \arrow[r] \arrow[u] & \Sigma^5 R \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Here $F_1 = \Sigma\mathcal{A}(1) \oplus \Sigma^4 F$. It is easy to check that the top and bottom rows in the preceding diagram are each the unique non-trivial extension.

Applying this procedure again, we get

$$\begin{tikzcd}
0 \arrow[r] & M_2 \arrow[r] & P_2 \arrow[r] & \Sigma^4 R \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^2\mathcal{A}(1) \arrow[r] \arrow[u] & F_2 \arrow[r] \arrow[u] & \Sigma^4 F \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Sigma^9 DM_1 \arrow[r] \arrow[u] & \Sigma P_3 \arrow[r] \arrow[u] & \Sigma^5 R \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

Here $F_2 = \Sigma^2\mathcal{A}(1) \oplus \Sigma^4 F$ and $DM = \mathrm{Hom}(M, \mathbf{F}_2)$ is the dual of M. This is not the short exact sequence involving P_3 which we have claimed. In fact, the interaction between the upside down question mark $\Sigma^9 DM_1$ and the $\Sigma^4\mathcal{A}(1)/\!\!/\mathcal{A}(0)$ in $\Sigma^5 R$ is non-trivial, producing $M_3 = \Sigma^3\mathcal{A}(1)/(Sq^2Sq^2Sq^2)$. We obtain a more useful description of P_3 by taking this into account. In the following diagram, the middle two rows are a map of short exact sequences and the snake lemma identifies the vertical kernels and cokernels.

$$\begin{tikzcd}
 & & \Sigma^8 R \arrow[r, equal] & \Sigma^8 R & \\
0 \arrow[r] & \Sigma^8 DM_1 \arrow[r] & P_3 \arrow[r] \arrow[u] & \Sigma^4 R \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Sigma^4\mathcal{A}(1)/\!\!/\mathcal{A}(0) \arrow[r] \arrow[u] & \Sigma^3\mathcal{A}(1) \arrow[r] \arrow[u] & \Sigma^3\mathcal{A}(1)/\!\!/\mathcal{A}(0) \arrow[r] \arrow[u] & 0 \\
 & \Sigma^9\mathbf{F}_2 \arrow[r, equal] \arrow[u] & \Sigma^9\mathbf{F}_2 \arrow[u] & &
\end{tikzcd}$$

Factoring the middle vertical exact sequence into short exact sequences, and taking the second half, we obtain the short exact sequence we want:

$$0 \longrightarrow M_3 \longrightarrow P_3 \longrightarrow \Sigma^8 R \longrightarrow 0.$$

Taking free resolutions of each of these and taking kernels yields the last claimed short exact sequence:

$$
\begin{tikzcd}
0 \arrow[r] & M_3 \arrow[r] & P_3 \arrow[r] & \Sigma^8 R \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^3 \mathcal{A}(1) \arrow[r] \arrow[u] & F_3 \arrow[r] \arrow[u] & \Sigma^8 F \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \Sigma^9 \mathbf{F}_2 \arrow[r] \arrow[u] & \Sigma P_4 \arrow[r] \arrow[u] & \Sigma^9 R \arrow[r] \arrow[u] & 0
\end{tikzcd}
$$

where $F_3 = \Sigma^3 \mathcal{A}(1) \oplus \Sigma^8 F$. Again the top and bottom rows are each the unique non-trivial extension.

Finally, the minimal free module mapping onto $\Sigma^8 R$ factors through the latter map $P_4 \longrightarrow \Sigma^8 R$ producing the diagram

$$
\begin{tikzcd}
 & \Sigma^8 \mathbf{F}_2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & 0 \\
0 \arrow[r] & \Sigma^8 \mathbf{F}_2 \arrow[r] \arrow[u, equal] & P_4 \arrow[r] \arrow[u] & \Sigma^8 R \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & \Sigma^8 F \arrow[r, equal] \arrow[u] & \Sigma^8 F \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u, equal] & \Sigma P_5 \arrow[r] \arrow[u] & \Sigma^9 R \arrow[u] \arrow[uuull, bend right=60]
\end{tikzcd}
$$

The ker-coker sequence above is the 9-fold suspension of our key sequence (1), which shows that $\Omega^4 P = \Sigma^4 P_5 = \Sigma^{12} P$. $\square$

Remark 4.5. *The modules P_i occur at the top of page 419 in Margolis ([7]) in connection with the study of Q_1-localizations of $\mathbf{F}_2$. The periodicity, $\Omega^4 P_n \simeq \Sigma^{12} P_n$ is special to $\mathcal{A}(1)$. Over $\mathcal{A}(n)$, $n > 1$, the Q_1-localization of $\mathbf{F}_2$ has Ω-period 2^n ([7, Theorem 23.12]).*

For reference, in Figure 3 we record the exact sequence which we obtain by composing the four short exact sequences we have just found. From this, we may construct the minimal free resolution of P by extending it by periodicity. Observe that it contains within it resolutions of all the P_n. A rendering of this as art can be found at [13].

5. The free summand in $P^{\otimes(n)}$

We have now shown that

$$P^{\otimes n} = P_n \oplus F_n$$

where F_n is a free $\mathcal{A}(1)$-module. We can therefore give a complete decomposition of $P^{\otimes(n)}$ by simply computing the Hilbert series of the free part. This can be found in Yu's thesis ([12, Theorem 4.2]). The most transparent form of the Hilbert series for the P_n can simply be read off from Theorem 4.3.

Lemma 5.1. *$H(P_{4k+i}) = t^{8k} H(P_i)$ and*

- $H(P_0) = \dfrac{t^{-1}}{1-t}$
- $H(P_1) = \dfrac{t^{1}}{1-t}$

$$
\begin{tikzcd}[column sep=small]
& \vdots \arrow[d] \\
& \Sigma^{13}\mathcal{A}(1) \oplus \bigoplus_{i\geq 0} \Sigma^{4i+15}\mathcal{A}(1) \arrow[dl] \arrow[dd, "{\begin{bmatrix} Sq^2Sq^1 & Sq^2Sq^1Sq^2 & 0 & \cdots \\ 0 & Sq^1 & Sq^2Sq^1Sq^2 & \cdots \\ 0 & 0 & Sq^1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}"] \\
\Sigma^{12}P \arrow[dr, tail] & \\
& \bigoplus_{i\geq 0} \Sigma^{4i+10}\mathcal{A}(1) \arrow[dl] \arrow[dd, "{\begin{bmatrix} Sq^2Sq^1Sq^2 & 0 & \cdots \\ Sq^1 & Sq^2Sq^1Sq^2 & \cdots \\ 0 & Sq^1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}}"] \\
\Sigma^3 P_4 \arrow[dr, tail] & \\
& \bigoplus_{i\geq 0} \Sigma^{4i+5}\mathcal{A}(1) \arrow[dl] \arrow[dd, "{\begin{bmatrix} Sq^2 & 0 & 0 & \cdots \\ Sq^1 & Sq^2Sq^1Sq^2 & 0 & \cdots \\ 0 & Sq^1 & Sq^2Sq^1Sq^2 & \cdots \\ 0 & 0 & Sq^1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}"] \\
\Sigma^2 P_3 \arrow[dr, tail] & \\
& \Sigma^3\mathcal{A}(1) \oplus \bigoplus_{i\geq 0} \Sigma^{4i+4}\mathcal{A}(1) \arrow[dl] \arrow[dd, "{\begin{bmatrix} Sq^2 & Sq^2Sq^1 & 0 & \cdots \\ 0 & Sq^1 & Sq^2Sq^1Sq^2 & \cdots \\ 0 & 0 & Sq^1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}"] \\
\Sigma P_2 \arrow[dr, tail] & \\
& \Sigma\mathcal{A}(1) \oplus \bigoplus_{i\geq 0} \Sigma^{4i+3}\mathcal{A}(1) \arrow[d, two heads] \\
& P_1 = P \arrow[d] \\
& 0
\end{tikzcd}
$$

FIGURE 3. The minimal $\mathcal{A}(1)$ resolution of $P = P_1$, showing one full cycle and one term of the next. Diagonal dots indicate continuation of the column containing $Sq^2Sq^1Sq^2$ and Sq^1 down the diagonal.

- $H(P_2) = \dfrac{t^2}{1-t} + t^3 + t^5 + t^6$
- $H(P_3) = \dfrac{t^3}{1-t} + t^6 + t^7$

Another form works a bit better in connection with the Hilbert series for $P^{\otimes(n)}$.

Lemma 5.2. *The Hilbert series*

$$H(P_n) = \frac{t^{2n}}{1-t} Q_n(t)$$

where

$$Q_n(t) = \begin{cases} \dfrac{1}{t} & n \equiv 0,1 \pmod 4 \\ \dfrac{1+t-t^2+t^3-t^5}{t^2} & n \equiv 2 \pmod 4 \\ \dfrac{1+t^3-t^5}{t^3} & n \equiv 3 \pmod 4 \end{cases}$$

Proof. Straightforward. □

We can now locate the summands in the free parts F_n.

Theorem 5.3. *The Hilbert series of the modules F_n are*

$$H(F_n) = H(\mathcal{A}(1)) \frac{t^n(1-t^n(1-t)^{n-1}Q_n(t))}{(1-t)^{n-1}(1-t^4)(1+t^3)}$$

Proof. We simply compute

$$\begin{aligned} \frac{H(P^{\otimes n}) - H(P_n)}{H(\mathcal{A}(1))} &= \frac{\left(\dfrac{t}{1-t}\right)^n - \dfrac{t^{2n}}{1-t} Q_n}{(1+t)(1+t^2)(1+t^3)} \\ &= \frac{t^n - t^{2n}(1-t)^{n-1}Q_n(t)}{(1-t)^n(1+t)(1+t^2)(1+t^3)} \\ &= \frac{t^n(1-t^n(1-t)^{n-1}Q_n(t))}{(1-t)^{n-1}(1-t^4)(1+t^3)} \end{aligned}$$

□

The following special cases are of particular interest, and are the correct replacement for Lemma 2 in [8], where the free part of $P \otimes P$ is asserted to be $\mathcal{A}(1) \otimes \Sigma^2 \mathbf{F}_2[u_2, v_4]$.

Corollary 5.4. *As $\mathcal{A}(1)$-modules*

$$P \otimes P_0 \ \cong\ P \oplus (\mathcal{A}(1) \otimes \mathbf{F}_2[u_2, v_4])$$

and

$$P \otimes P \ \cong\ P_2 \oplus \bigoplus_{\substack{i,j \geq 0 \\ i+j>0}} \Sigma^{4i+4j} \mathcal{A}(1) \oplus \bigoplus_{i,j \geq 0} \Sigma^{4i+4j+6} \mathcal{A}(1)$$

Remark 5.5. *P_0 is the homology of $T(-\lambda)$, the Thom complex of the negative of the line bundle over $P = BC_2$. As a consequence, the first isomorphism in Theorem 5.4 can be used to give a homotopy equivalence*

$$ko \wedge BC_2 \wedge T(-\lambda) \ \simeq\ (ko \wedge BC_2) \vee H\mathbf{F}_2[u_2, v_4]$$

6. Locating P_n in $P^{\otimes(n)}$

The method of the preceding sections does not give us explicit embeddings $P_n \longrightarrow P^{\otimes(n)}$. We produce these now (when $n > 0$). There are choices involved, but the inductive determination of the isomorphism type also gives us a way to inductively find P_{n+1} as a summand of $P_n \otimes P \subset P^{\otimes(n)} \otimes P$, reducing the work dramatically.

Let us write $x_1^{i_1} \dots x_n^{i_n}$ as $i_1 \dots i_n$ and define $\overline{i_1 \dots i_n}$ to be the orbit sum of $i_1 \dots i_n$.

Theorem 6.1. *P_n can be embedded in $P^{\otimes(n)}$ as follows:*

- $P_1 = P$
- $M_2 = \langle 11, \overline{12}, 22, \overline{14}, \overline{24}\rangle$
- $P_2 = M_2 + \langle 21, 4i \mid i \geq 1\rangle$
- $M_3 = \langle 111, \overline{112}, \overline{122}, 222, 222 + \overline{114}, \overline{124}, \overline{224}\rangle$
- $P_3 = M_3 + \langle 124 + 142 + 421, 44i \mid i \geq 1\rangle$
- $M_4 = \langle 2222 + \overline{1124}\rangle$
- $P_4 = M_4 + \langle 2221 + \overline{1114}, \overline{122}4 + \overline{124}2, 2224, \overline{224}i \mid i \geq 1\rangle$
- $P_{n+4} \;\cong\; \langle 2222 + \overline{1124}\rangle \otimes P_n$.

Remark 6.2. *There are several notable points about these submodules.*

1. *The first three generators x_1, x_1x_2, and $x_1x_2x_3$, are obvious from the connectivity: the connectivity of P_n is n for $n < 4$.*
2. *The fourth, $x_1^2x_2^2x_3^2x_4 + \overline{x_1x_2x_3x_4^4}$ in degree 7, is less so. The classes of degrees less than 7 all lie in free summands of $P_3 \otimes P$. Modulo those free summands, there are 4 possible choices for the degree 7 class in P_4:*
$$2221 + \overline{1114} + \alpha_0(2221 + \overline{221}2) + \alpha_1(1114 + \overline{112}3)$$
for $\alpha_i \in \{0, 1\}$.
3. *Applying Sq^1 to any of these four classes yields the same 'periodicity class' $B = 2222 + \overline{1124}$ in degree 8. From $H(P, Q_1) = [x_1^2]$, we know that $H(P_4, Q_1) = [x_1^2x_2^2x_3^2x_4^2]$, but since $Sq^2(x_1^2x_2^2x_3^2x_4^2) \neq 0$, the 'periodicity class' must have additional terms, which turn out to be exactly $Q_0Q_1(x_1x_2x_3x_4)$, or $\overline{1124}$ in our abbreviated notation.*
4. *Above the bottom few degrees, each of the P_i can be written as the tensor product of an $\mathcal{A}(1)$-annihilated class with P. These $\mathcal{A}(1)$-annihilated classes are B^i, $x_1^4B^i$, $x_1^4x_2^4B^i$, and $\overline{x_1^2x_2^2x_3^4}B^i$.*

Proof. Evidently $P_1 = P$. For P_2 and P_3, it is a simple matter to verify that x_1x_2 and $x_1x_2x_3$ generate M_2 and M_3, respectively and that they are the only choices. To finish P_2, we must choose a class x with $Sq^1x = x_1^2x_2^2 \in M_2$ and $(\langle x\rangle + M_2)/M_2 \;\cong\; \Sigma^3\mathcal{A}(1)/\!\!/\mathcal{A}(0)$. Clearly $x = x_1^2x_2$ serves, with the rest of P_2 then given by $x_1^4(x_2^i)$.

For P_3, we need x with $Sq^1x = \overline{x_1^2x_2^2x_3^4}$. By choosing $x = x_1x_2^2x_3^4 + x_1x_2^4x_3^2 + x_1^4x_2^2x_3$, the rest of P_3 is given by $x_1^4x_2^4(x_3^i)$.

For P_4, we need a class in degree 7 in $P_3 \otimes P$ which is not in $\mathrm{Im}(Sq^1) + \mathrm{Im}(Sq^2)$ and whose annihilator ideal is (Sq^2Sq^1). Solving $Sq^1x \neq 0$, $Sq^2Sq^1x = 0$, $Sq^2Sq^1Sq^2x \neq 0$, for $x \notin \mathrm{Im}(Sq^1) + \mathrm{Im}(Sq^2)$, we arrive at the 4 choices in Remark 6.2.(2) above. Our choice, $\alpha_0 = \alpha_1 = 0$, gives the version of P_4/M_4 which is simplest to describe.

Finally, consider periodicity. Since M_4 is a trivial $\mathcal{A}(1)$ module, tensoring with it is the same as 8-fold suspension. Now, if we tensor the short exact sequence $0 \longrightarrow M_4 \longrightarrow P_4 \longrightarrow \Sigma^8R \longrightarrow 0$ with P_n, we get
$$0 \longrightarrow M_4 \otimes P_n \longrightarrow P_4 \otimes P_n \longrightarrow \Sigma^8 R \otimes P_n \longrightarrow 0.$$
The Kunneth theorem and Theorem 2.7 imply that $M_4 \otimes P_n$ is stably isomorphic to $P_4 \otimes P_n$, and hence to $P^{\otimes(4)} \otimes P^{\otimes(n)}$. Since $M_4 \otimes P_n$ is indecomposable, it follows that it is isomorphic to P_{n+4} and that the inclusion $M_4 \otimes P_n \subset P_4 \otimes P_n \subset P^{\otimes(4)} \otimes P^{\otimes(n)}$ serves our purpose. $\square$

7. Q_i-local $A(1)$-modules

The modules P_n are Q_1-local, in the sense that $H(P_n, Q_i) = 0$ unless $i = 1$. Similarly, R is Q_0-local. In addition, Theorem 4.3 and Corollary 3.3 say that the modules P_0 and ΣR are stably idempotent. This

implies that tensoring with either of these is a stably idempotent functor. The Künneth isomorphism gives

$$H(P_0 \otimes M, Q_0) = 0 \otimes H(M, Q_0) = 0 \qquad \text{and} \qquad H(\Sigma R \otimes M, Q_1) = 0 \otimes H(M, Q_1) = 0,$$

which motivates the consideration of the following subcategories of the category of $\mathcal{A}(1)$ or $E(1)$-modules.

Definition 7.1. *Let B-$\mathrm{Mod}^{(k)}$ be the full subcategory of B-Mod^f containing the Q_k-*local *modules, i.e., those whose homology with respect to Q_i is 0 unless $i = k$.*

Direct sum and tensor product make B-Mod^f and B-$\mathrm{Mod}^{(k)}$ into commutative semi-rings, and the idempotence of ΣR and P_0 make tensoring with them into semi-ring homomorphisms, up to stable equivalence.

Definition 7.2. *Let $F_i : B\text{-}\mathrm{Mod}^f \longrightarrow B\text{-}\mathrm{Mod}^{(i)}$ by*

- *$F_0(M) = \Sigma R \otimes M$, and*
- *$F_1(M) = P_0 \otimes M$.*

In a notation intended to be suggestive, we also make the following definition.

Definition 7.3. *Let $F_{i/4} : B\text{-}\mathrm{Mod}^f \longrightarrow B\text{-}\mathrm{Mod}^{(1)}$ by*

$$F_{i/4}(M) = \Sigma^{-2i} P_i \otimes M$$

Note that the $F_{i/4}$ give fourth roots of F_1. We have the slightly unfortunate situation that $F_{0/4}$ is F_1 rather than F_0, but if we think of the subscripts $i/4$ as specifying fourth roots of 1, the notation makes sense.

We record the basic properties of these functors.

Theorem 7.4. *F_0 and F_1 are exact. The functors they induce on stable module categories are idempotent, orthogonal, semi-ring homomorphisms. There are short exact sequences*

$$0 \longrightarrow M \xrightarrow{\eta_M} F_1(M) \longrightarrow \Sigma^{-1} F_0(M) \longrightarrow 0$$

and

$$0 \longrightarrow \Sigma^3 F_{1/4}(M) \longrightarrow F_0(M) \xrightarrow{\epsilon_M} M \longrightarrow 0.$$

In addition,

- *$H(F_i(M), Q_i) = H(M, Q_i)$ and $H(F_i(M), Q_j) = 0$ if $j \neq i$.*
- *$H(F_{i/4}(M), Q_0) = 0$ and $H(F_{i/4}(M), Q_1) = H(M, Q_1)$.*
- *$F_0 F_{i/4} \simeq 0 \simeq F_{i/4} F_0$*
- *$F_{i/4} F_{j/4} \simeq F_{(i+j)/4}$.*
- *$F_1 = F_{4/4}$.*
- *$F_{(i+4)/4} = F_{i/4}$.*
- *If $M \in B$-$\mathrm{Mod}^{(0)}$ then $\epsilon_M : F_0(M) \longrightarrow M$ is a stable equivalence and $F_1(M) \simeq 0$.*
- *If $M \in B$-$\mathrm{Mod}^{(1)}$ then $\eta_M : M \longrightarrow F_1(M)$ is a stable equivalence and $F_0(M) \simeq 0$.*

Proof. The fact that F_0 and F_1 are idempotent and preserve tensor products (up to stable equivalence) follows from the idempotence of the modules ΣR and P_0. Their exactness follows from that of tensoring over $\mathbf{F}_2$. Their orthogonality follows from the fact that $\Sigma R \otimes P_0$ is free. Similarly, $\Sigma R \otimes P_i$ free implies $F_0 F_{i/4}$ is stably trivial. Additivity of the $F_{i/4}$ in i is simply the relation $\Sigma^{-2i} P_i \otimes \Sigma^{-2j} P_j \simeq \Sigma^{-2(i+j)} P_{i+j}$. Since $\Sigma^{-8} P_{i+4} = P_i$, we get $F_{(i+4)/4} = F_{i/4}$. The exact sequences follow from the short exact sequences of modules

$$0 \longrightarrow \mathbf{F}_2 \xrightarrow{\eta} P_0 \longrightarrow R \longrightarrow 0$$

and

$$0 \longrightarrow \Sigma P_1 \longrightarrow \Sigma R \xrightarrow{\epsilon} \mathbf{F}_2 \longrightarrow 0.$$

When $M \in B$-$\mathrm{Mod}^{(0)}$, the Q_1-homology of M is 0. Hence $F_{i/4}(M) \simeq 0$ and ϵ_M is a stable equivalence. Similarly, if $M \in B$-$\mathrm{Mod}^{(1)}$ then $F_0(M) \simeq 0$ and hence η_M is a stable equivalence. □

It is worth emphasizing the consequences of the last two statements.

Corollary 7.5. *The functors F_i split the inclusions, up to stable equivalence: the composites*

$$B\text{-Mod}^{(i)} \longrightarrow B\text{-Mod}^{f} \xrightarrow{F_i} B\text{-Mod}^{(i)}$$

are stably equivalent to the identity. Precisely,

- *If M is in $B\text{-Mod}^{(0)}$ then $\Sigma^3 F_{1/4}(M)$ is a free $\mathcal{A}(1)$-module and*
$$F_0(M) = \Sigma R \otimes M \;\cong\; M \oplus \Sigma^3 F_{1/4}(M).$$
- *If M is in $B\text{-Mod}^{(1)}$ then $\Sigma^{-1} F_0(M)$ is a free $\mathcal{A}(1)$-module and*
$$F_1(M) = P_0 \otimes M \;\cong\; M \oplus \Sigma^{-1} F_0(M).$$

Finally, the fundamental short exact sequence (1) allows us to give a criterion for stable equivalence in $B\text{-Mod}$, not just $B\text{-Mod}^f$.

Corollary 7.6. *A homomorphism $f : M \longrightarrow N$ in $B\text{-Mod}$ is a stable equivalence iff both $F_0(f)$ and $F_1(f)$ are stable equivalences.*

Proof. Evident from the short exact sequences in the preceding proposition. □

8. Pic and $\mathrm{Pic}^{(k)}$

Again let B be either $E(1)$ or $\mathcal{A}(1)$ and $\mathrm{St}(\mathcal{C})$ the stable category of a subcategory $\mathcal{C}$ of $B\text{-Mod}$ (Definition 2.4).

Definition 8.1. *Let $\widetilde{\mathrm{Pic}}(B)$ and $\mathrm{Pic}^{(k)}(B)$ be the multiplicative groups in $\mathrm{St}(B\text{-Mod}^f)$ and $\mathrm{St}(B\text{-Mod}^{(k)})$, respectively. Let $\mathrm{Pic}(B)$ be the subgroup of $\widetilde{\mathrm{Pic}}(B)$ whose elements are represented by finitely generated modules.*

It would be interesting to know whether $\mathrm{Pic}(B) = \widetilde{\mathrm{Pic}}(B)$, and, if not, how much larger $\widetilde{\mathrm{Pic}}(B)$ is.

Adams and Priddy characterize the elements in $\mathrm{Pic}(B)$. It is pertinent to recall that $H(M, Q_i)$ depends only upon the stable isomorphism type of M.

Lemma 8.2. [2, Lemma 3.5] *$M \in \mathrm{Pic}(B)$ iff each $H(M, Q_i)$ is one dimensional.*

Adams and Priddy remark that, if one drops the hypothesis of finite generation, then having $H(M, Q_i)$ one dimensional for each i no longer implies that M is invertible. The module $P_0 \oplus \Sigma R$ is an example. One direction does hold in general, though.

Lemma 8.3. *If $M \in \mathrm{Pic}^{(k)}(B)$ then $H(M, Q_k)$ is one dimensional.*

The converse, Corollary 8.11, will follow from the calculations of $\mathrm{Pic}^{(k)}$ in this section, since those calculations will show that if $M \in B\text{-Mod}^{(k)}$ and $H(M, Q_k)$ is one dimensional, then M is stably isomorphic to an invertible module.

It is not *a priori* obvious that $\mathrm{Pic}^{(k)}$ is nonempty. However, we can see that it is, by noting that it has a unit.

Corollary 8.4. *ΣR is the unit for tensor product in $\mathrm{St}(B\text{-Mod}^{(0)})$ and P_0 is the unit for tensor product in $\mathrm{St}(B\text{-Mod}^{(1)})$.*

Proof. This is just a restatement of Corollary 7.5. □

After characterizing the invertible B-modules, Adams and Priddy go on to compute $\mathrm{Pic}(E(1))$ and $\mathrm{Pic}(\mathcal{A}(1))$.

Theorem 8.5. [2, Theorem 3.6] $\mathrm{Pic}(E(1)) = \mathbf{Z} \oplus \mathbf{Z}$*, generated by $\Sigma \mathbf{F}_2$ and the augmentation ideal* $\Omega \mathbf{F}_2 = \mathrm{Ker}(E(1) \longrightarrow \mathbf{F}_2)$.

Theorem 8.6. [2, Theorem 3.7] $\mathrm{Pic}(\mathcal{A}(1)) = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/(2)$*, generated by $\Sigma \mathbf{F}_2$, the augmentation ideal* $\Omega \mathbf{F}_2 = \mathrm{Ker}(\mathcal{A}(1) \longrightarrow \mathbf{F}_2)$*, and $\Sigma^{-4} M_2$.*

The module $J = \Sigma^{-4} M_2$ is known as the 'joker' for its role as a torsion element in $\mathrm{Pic}(\mathcal{A}(1))$ and for the resemblance of its diagrammatic depiction (Figure 1) to a traditional jester's hat.

In his thesis ([12]), Cherng-Yih Yu computed $\mathrm{Pic}^{(1)}$ for both $E(1)$ and $\mathcal{A}(1)$. We give the calculation of $\mathrm{Pic}^{(1)}(E(1))$ here because it is brief and serves as a model for our calculations of $\mathrm{Pic}^{(0)}$.

Recall that as an $E(1)$-module, $P_i \simeq \Sigma^{2i} P_0$ (see Remark 3.4).

Proposition 8.7. [12, Lemma 2.5] *If $M \in E(1)\text{-Mod}^{(1)}$ and $H(M, Q_1) = \Sigma^s \mathbf{F}_2$ then $M \simeq \Sigma^s P_0$. Therefore, $\mathrm{Pic}^{(1)}(E(1)) = \{\Sigma^i P_0\} \cong \mathbf{Z}$.*

Proof. By suspending appropriately, we may assume that $M \in E(1)\text{-Mod}^{(1)}$ and $H(M, Q_1) = \mathbf{F}_2$. We may also assume that M is reduced. Let $0 \neq \langle [x] \rangle = H(M, Q_1)$, so that $Q_1(x) = 0$ and $x \notin \mathrm{Im}(Q_1)$. There are two possibilities:

(1) $Sq^1 x \neq 0$
(2) $Sq^1 x = 0$

In the first case, $Q_1 Sq^1 x = 0$ because M is reduced, so $Sq^1 x = Q_1 x_1$ for some x_1. (This because $[x]$ is the only nonzero Q_1 homology class of M.) Again, M reduced implies that $x_1 \notin \mathrm{Im}(Sq^1)$, so that $Sq^1 x_1 \neq 0$. Again, we cannot have $Q_1 Sq^1 x_1 \neq 0$, so that $Sq^1 x_1 = Q_1 x_2$ for some x_2. Continuing in this way, it follows by induction that M is not bounded-below, contrary to our assumption.

Therefore, we must have $Sq^1 x = 0$. Then $x = Sq^1 x_0$ for some x_0 and $x_0 \notin \mathrm{Im}(Q_1)$ because M is reduced. Hence $Q_1 x_0 \neq 0$. Again, the fact that M is reduced implies that $Q_1 x_0 = Sq^1 x_1$ for some x_1. For induction, we may suppose that we have found a sequence of elements x_i such that $Q_1 x_{i-1} = Sq^1 x_i \neq 0$, for $0 \leq i \leq n$. Then, since M is reduced, $x_n \notin \mathrm{Im}(Q_1)$, so $Q_1 x_n \neq 0$ and there must be x_{n+1} such that $Q_1 x_{n+1} = Sq^1 x_n$.

The submodule of M generated by the x_i is isomorphic to P_0 and the inclusion $P_0 \longrightarrow M$ induces an isomorphism of Q_i homologies, hence is a stable isomorphism.

Now suppose that $M \in \mathrm{Pic}^{(1)}(E(1))$. By Lemma 8.3, $H(M, Q_1) = \Sigma^s \mathbf{F}_2$ for some s, and therefore $M \simeq \Sigma^s P_0$. Finally, observe that the $\Sigma^i P_0$ are all distinct because $H(\Sigma^i P_0, Q_1) = \Sigma^i \mathbf{F}_2$. $\square$

The case of $\mathcal{A}(1)$ is much more intricate.

Theorem 8.8. [12, Theorem 2.1] *If $M \in \mathcal{A}(1)\text{-Mod}^{(1)}$ and $H(M, Q_1) = \Sigma^s \mathbf{F}_2$ then $M \simeq \Sigma^{s-2b} P_b$ for some b. Therefore, $\mathrm{Pic}^{(1)}(\mathcal{A}(1)) = \{\Sigma^i P_n\} \cong \mathbf{Z} \oplus \mathbf{Z}/(4)$ with $(a, b) \in \mathbf{Z} \oplus \mathbf{Z}/(4)$ corresponding to $\Sigma^{a-2b} P_b$.*

The proof of this Theorem takes a substantial amount of work, occupying 12 pages in [12], and I have no major simplifications to offer.

For $E(1)$, the determination of $\mathrm{Pic}^{(0)}$ is similar to that of $\mathrm{Pic}^{(1)}$.

Proposition 8.9. *If $M \in E(1)\text{-Mod}^{(0)}$ and $H(M, Q_0) = \Sigma^s \mathbf{F}_2$ then $M \simeq \Sigma^{s+1} R$. Therefore, $\mathrm{Pic}^{(0)}(E(1)) = \{\Sigma^i R\} \cong \mathbf{Z}$.*

Proof. By suspending appropriately, we may assume that $M \in E(1)\text{-Mod}^{(0)}$ and $H(M, Q_0) = \Sigma^{-1} \mathbf{F}_2$. We may also assume that M is reduced.

Let $0 \neq \langle [x] \rangle = H(M, Q_0)$, so that $Sq^1 x = 0$ and $x \notin \mathrm{Im}(Sq^1)$. There are two possibilities:

(1) $Q_1 x = 0$
(2) $Q_1 x \neq 0$

In the first case, $x = Q_1 y_0$ for some y_0, which cannot be in the image of Sq^1, since M is reduced, so that $Sq^1 y_0 \neq 0$. We may then assume for induction that we are given y_i such that $Q_1 y_i = Sq^1 y_{i-1}$ for $0 \leq i \leq n$, and such that $Q_1 y_0 = x$ and $Sq^1 y_n \neq 0$. The assumption that M is reduced allows us to extend this another step, completing the induction. We conclude that M is not bounded-below, contrary to assumption.

It therefore follows that $Q_1 x \neq 0$. Then $Sq^1 Q_1 x = 0$ because M is reduced, so $Q_1 x = Sq^1 x_1$ for some x_1. Again, M reduced implies that $Q_1 x_1 \neq 0$. We may assume for induction that we have elements x_i with $Sq^1 x_i = Q_1 x_{i-1} \neq 0$ for $0 \leq i \leq n$ and $Q_1 x_n \neq 0$. (Let $x_0 = x$ here.) Then M reduced implies $Q_1 x_n = Sq^1 x_{n+1}$ for some x_{n+1} and $Q_1 x_{n+1} \neq 0$, completing the induction. The x_i generate a submodule isomorphic to R and the inclusion $R \longrightarrow M$ induces a stable isomorphism.

Now suppose that $M \in \mathrm{Pic}^{(0)}(E(1))$. By Lemma 8.3, $H(M, Q_0) = \Sigma^s \mathbf{F}_2$ for some s, and therefore $M \simeq \Sigma^{s+1} R$. Finally, observe that the $\Sigma^i R$ are all distinct because $H(\Sigma^i R, Q_0) = \Sigma^{i-1} \mathbf{F}_2$. $\square$

For $\mathcal{A}(1)$, the argument is a bit more complicated, but the conclusion is the same.

Proposition 8.10. *If $M \in \mathcal{A}(1)\text{-Mod}^{(0)}$ and $H(M, Q_0) = \Sigma^s \mathbf{F}_2$ then $M \simeq \Sigma^{s+1} R$. Therefore, $\mathrm{Pic}^{(0)}(\mathcal{A}(1)) = \{\Sigma^i R\} \cong \mathbf{Z}$.*

Proof. By suspending appropriately, we may assume that $M \in \mathcal{A}(1)\text{-Mod}^{(0)}$ and $H(M, Q_0) = \Sigma^{-1}\mathbf{F}_2$. We may also assume that M is reduced: $Sq^2 Sq^2 Sq^2$ acts as 0 on M. Let $0 \neq \langle [x] \rangle = H(M, Q_0)$, so that $Sq^1 x = 0$ and $x \notin \mathrm{Im}(Sq^1)$.

Let $M \cong M_0 \oplus M_1$ as an $E(1)$-module, where M_0 is a reduced $E(1)$-module and M_1 is $E(1)$-free. Then M_0 is in $\mathrm{Pic}^{(0)}(E(1))$ with $H(M_0, Q_0) = H(M, Q_0) = [x]$. By the preceding Proposition, $M_0 \cong R$. In particular, $Q_1 x \neq 0$. Since $Sq^1 x = 0$, $Q_1 x \neq 0$ implies that $Sq^1 Sq^2 x \neq 0$. There are two possibilities:

(1) $Sq^2 Sq^1 Sq^2 x \neq 0$
(2) $Sq^2 Sq^1 Sq^2 x = 0$

In the first case, the submodule $\langle x \rangle$ is $\Sigma^{-1}\mathcal{A}(1) /\!\!/ \mathcal{A}(0)$ since M is reduced and $Sq^1 x = 0$. The long exact sequences in Q_i-homology induced by the short exact sequence

$$0 \longrightarrow \langle x \rangle \longrightarrow M \longrightarrow M/\langle x \rangle \longrightarrow 0$$

imply that $H(M/\langle x \rangle, Q_1) = 0$ and $H(M/\langle x \rangle, Q_0) = \langle [y] \rangle$ with $Q_0 y = Sq^2 Sq^1 Sq^2 x$. Then $M/\langle x \rangle$ satisfies the same hypotheses as M shifted up by 4 degrees. We can thus inductively construct $R \longrightarrow M$ inducing an isomorphism in Q_0 and Q_1 homology. Hence M is stably isomorphic to R as claimed.

The second alternative implies that the submodule generated by x is spanned by x, $Sq^2 x$ and $Sq^1 Sq^2 x$. This has Q_1 homology $\langle [Sq^2 x] \rangle$. The long exact homology sequence for

$$0 \longrightarrow \langle x \rangle \longrightarrow M \longrightarrow M/\langle x \rangle \longrightarrow 0$$

then implies that $H(M/\langle x \rangle, Q_0) = 0$ and $H(M/\langle x \rangle, Q_1) = \langle [y] \rangle$ with $Q_1 y = Sq^2 x$. By Yu's theorem (Theorem 8.8), $M/\langle x \rangle$ must be a suspension of P_n for some n. (It is actually isomorphic to $\Sigma^i P_n$, not just stably equivalent to it, because it is reduced, being a quotient of the reduced module M.) Further, if we let $y \in M$ be a class whose image in $M/\langle x \rangle$ generates $H(M/\langle x \rangle, Q_1)$ then $Q_1 y = Sq^2 x$. Now $Sq^1 y = 0$ because this is so in each P_n and because x, which is in the same degree as $Sq^1 y$, is not in the image of Sq^1. Thus, we must have $Sq^1 Sq^2 y = Sq^2 x$. This is impossible. In P_0, $Sq^2 y = 0$, while in P_n, $1 \leq n \leq 3$, $Sq^2 y$ is in the image of Sq^1. Since $\langle x \rangle$ is zero in this degree, the same holds in M. This contradiction shows that the second alternative does not happen, proving the theorem.

Now suppose that $M \in \mathrm{Pic}^{(0)}(\mathcal{A}(1))$. By Lemma 8.3, $H(M, Q_0) = \Sigma^s \mathbf{F}_2$ for some s, and therefore $M \simeq \Sigma^{s+1} R$. Finally, observe that the $\Sigma^i R$ are all distinct because $H(\Sigma^i R, Q_0) = \Sigma^{i-1}\mathbf{F}_2$. $\square$

From these last four results, we have the converse of Lemma 8.3.

Corollary 8.11. *A module $M \in B\text{-Mod}^{(k)}$ is in $\mathrm{Pic}^{(k)}(B)$ iff $H(M, Q_k)$ is one dimensional.*

It is useful to have explicit forms for these isomorphisms.

Corollary 8.12. *For $M \in \mathrm{Pic}^{(k)}(B)$, let $d_i(M)$ be defined by $H(M, Q_i) = \Sigma^{d_i(M)}\mathbf{F}_2$. Then $d_i : \mathrm{Pic}^{(k)}(B) \longrightarrow \mathbf{Z}$ is a homomorphism. It is an isomorphism if $i = 0$ or $B = E(1)$. When $i = 1$ and $B = \mathcal{A}(1)$, $\mathrm{Ker}(d_1) = \{\Sigma^{-2i} P_i\} \cong \mathbf{Z}/(4)$.*

Proof. The Künneth isomorphism implies that d_i is a homomorphism. The remainder follows directly from Theorem 8.8 and Propositions 8.7, 8.9, and 8.10. $\square$

When $i = 1$ and $B = \mathcal{A}(1)$ we need further invariants to detect $\mathrm{Ker}(d_1)$. It is possible to do this using invariants of the stable isomorphism type, by looking at $\mathrm{Ext}^1_{\mathcal{A}(1)}(M, \mathbf{F}_2)$, but they are cumbersome to define, so we content ourselves with an invariant defined assuming that the module is reduced.

Proposition 8.13. *If M is a reduced module in $\mathrm{Pic}^{(1)}(\mathcal{A}(1))$, let*

- *c be the connectivity (bottom non-zero degree) of M,*
- *$e = \dim(Sq^2(M_c))$, and*
- *$f = \dim(Sq^2 Sq^2(M_c))$.*

(Here $\dim$ refers to dimension as an $\mathbf{F}_2$ vector space.) Then $M \simeq \Sigma^{d_1(M)-2n} P_n$, where $n = d_1(M) - c - e + f$.

Proof. It is simplest to reverse engineer this. We compute these invariants for $\Sigma^i P_n$:

	$\Sigma^i P_0$	$\Sigma^i P_1$	$\Sigma^i P_2$	$\Sigma^i P_3$
c	i-1	i+1	i+2	i+3
d_1	i	i+2	i+4	i+6
e	1	0	1	1
f	0	0	1	1
$d_1 - c - e + f$	0	1	2	3

□

9. The homomorphisms from Pic to Pic$^{(k)}$

Restricting the functors F_i to Picard groups gives us embeddings of Pic into the localized Picard groups.

Proposition 9.1. *Each $F_k : \mathrm{Pic}(E(1)) \longrightarrow \mathrm{Pic}^{(k)}(E(1))$ is an epimorphism. Their product F, mapping $\mathrm{Pic}(E(1))$ to $\mathrm{Pic}^{(0)}(E(1)) \oplus \mathrm{Pic}^{(1)}(E(1))$, is a monomorphism with cokernel $\mathbf{Z}/(2)$. With respect to the basis $\{\Sigma F_2, \Omega F_2\}$ of Pic we have*

$$\begin{tikzcd}
\mathrm{Pic}(E(1)) \arrow[d, "F"] \arrow[dr, "\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}"] & \\
\mathrm{Pic}^{(0)}(E(1)) \oplus \mathrm{Pic}^{(1)}(E(1)) \arrow[r, "d_0 \oplus d_1"'] & \mathbf{Z} \oplus \mathbf{Z}
\end{tikzcd}$$

Proof. Explicitly, $F(M) = (F_0(M), F_1(M)) = (\Sigma R \otimes M, P_0 \otimes M)$. We simply compute:

$$d_0(\Sigma R \otimes \Sigma \mathbf{F}_2) = d_0(\Sigma^2 R) = 1$$

and

$$d_1(P_0 \otimes \Sigma \mathbf{F}_2) = d_1(\Sigma P_0) = 1,$$

while

$$d_0(\Sigma R \otimes \Omega \mathbf{F}_2) = d_0(\Omega \Sigma R) = d_0(\Sigma^2 R) = 1$$

and

$$d_1(P_0 \otimes \Omega \mathbf{F}_2) = d_1(\Omega P_0) = d_1(\Sigma^3 P_0) = 3.$$

□

Over $\mathcal{A}(1)$ we also have the torsion summands to consider.

Proposition 9.2. *The restriction maps $\mathrm{Pic}(\mathcal{A}(1)) \longrightarrow \mathrm{Pic}(E(1))$ and $\mathrm{Pic}^{(k)}(\mathcal{A}(1)) \longrightarrow \mathrm{Pic}^{(k)}(E(1))$ induce isomorphisms from the torsion free quotients of their domains to their codomains, and commute with F. Each $F_k : \mathrm{Pic}(\mathcal{A}(1)) \longrightarrow \mathrm{Pic}^{(k)}(\mathcal{A}(1))$ is an epimorphism. With respect to the basis $\{\Sigma F_2, \Omega F_2, J\}$ of Pic, the homomorphism $F : \mathrm{Pic}(\mathcal{A}(1)) \longrightarrow \mathrm{Pic}^{(0)}(\mathcal{A}(1)) \oplus \mathrm{Pic}^{(1)}(\mathcal{A}(1))$ is*

$$\begin{tikzcd}
\mathrm{Pic}(\mathcal{A}(1)) \arrow[d, "F"] \arrow[dr, "\begin{bmatrix} 1 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & \overline{1} & \overline{2} \end{bmatrix}"] & \\
\mathrm{Pic}^{(0)}(\mathcal{A}(1)) \oplus \mathrm{Pic}^{(1)}(\mathcal{A}(1)) \arrow[r] & \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}/(4)
\end{tikzcd}$$

with $\overline{k}$ denoting the coset $k + (4)$. The cokernel of F is $\mathbf{Z}/(4)$.

Proof. Again, we simply compute. The d_0 and d_1 calculations are the same as for $E(1)$. This implies the first claim and gives the upper left two by two submatrix. For the remainder, we first need to compute F_1. We have $F_1(\Sigma \mathbf{F}_2) = \Sigma P_0$, which projects to $\overline{0}$ in the $\mathbf{Z}/(4)$ summand. We also have $F_1(\Omega \mathbf{F}_2) = \Omega P_0 = \Sigma^3(\Sigma^{-2} P_1)$, which projects to $\overline{1}$ in the $\mathbf{Z}/(4)$ summand. Next,

$$d_0(F_0(J)) = d_0(\Sigma R \otimes J) = 0$$

$$d_1(F_1(J)) = d_1(P_0 \otimes J) = 0.$$

Finally, $P_0 \otimes J$ is stably isomorphic to $\Sigma^{-4}P_2$. To verify this, write

$$J = \Sigma^{-2}\mathcal{A}(1)/(Sq^3) = \{1, Sq^1, Sq^2, Sq^2Sq^1, Sq^2Sq^2\}.$$

Observe that $(P_0 \otimes J)^{\otimes(2)} \simeq P_0^{\otimes(2)} \otimes J^{\otimes(2)} \simeq P_0 \otimes \mathbf{F}_2 \simeq P_0$, so that $P_0 \otimes J$ is either P_0 or $\Sigma^{-4}P_2$. That it is the latter is confirmed by finding that the submodule spanned by $\{x^0 \otimes j \mid j \in J\}$ together with $\{x^{-1} \otimes Sq^2\} \cup \{x^{i+2} \otimes Sq^2 + x^i \otimes Sq^2Sq^2 \mid i \geq -1\}$ is isomorphic to $\Sigma^{-4}P_2$.

Determination of the cokernel is a simple Smith Normal Form calculation. □

10. Idempotents and localizations

Again let B be either $E(1)$ or $\mathcal{A}(1)$. In this section we show that F_0 and F_1 are essentially unique, in that the only stably idempotent modules in $B\text{-Mod}^f$ are the ones we have already seen.

Theorem 10.1. *If $M \in B\text{-Mod}^f$ is stably idempotent then M is one of 0, $\mathbf{F}_2$, P_0, ΣR, or $P_0 \oplus \Sigma R$.*

Proof. We give the proof for $B = \mathcal{A}(1)$. The proof for $E(1)$ is similar but easier.

We first note a simple fact: if $M \otimes M \simeq M$ then each $H(M, Q_i)$ must be either 0 or $\mathbf{F}_2$. This yields 4 possibilities.

If both are 0, then $0 \longrightarrow M$ is a stable equivalence by Theorem 2.7.

If exactly one Q_i-homology group is nonzero, we have the unit in $\mathrm{Pic}^{(i)}(\mathcal{A}(1))$, which must be either P_0 or ΣR by Theorems 8.8 and 8.10.

The final possibility is that $H(M, Q_0) = \mathbf{F}_2 = H(M, Q_1)$. In this case we tensor M with the short exact sequence

$$0 \longrightarrow \Sigma R \longrightarrow \mathbf{F}_2 \oplus \bigoplus_{i \geq 0} \Sigma^{4i-1}\mathcal{A}(1) \longrightarrow P_0 \longrightarrow 0 \tag{4}$$

whose extension cocycle is the inclusion $\Sigma P_1 \longrightarrow \Sigma R$. We get a short exact sequence

$$0 \longrightarrow F_0(M) \longrightarrow \widetilde{M} \longrightarrow F_1(M) \longrightarrow 0$$

in which $\widetilde{M} \simeq M$. By Theorem 7.4, each $F_i(M)$ is stably idempotent and Q_i-local. By the preceding paragraph, $F_0(M) \simeq \Sigma R$ and $F_1(M) \simeq P_0$. It remains to determine the possible extensions $\widetilde{M}$.

Using the resolution in Figure 3 we compute

$$\mathrm{Ext}^{1,0}_{\mathcal{A}(1)}(P_0, \Sigma R) = \mathbf{F}_2.$$

Therefore, the two possibilities are the split extension $M \simeq P_0 \oplus \Sigma R$ and the nonsplit $M \simeq \mathbf{F}_2$ in the sequence (4) above. □

11. Final remarks and a conjecture

As noted in 2.8, the detection of stable isomorphisms is more subtle in the category of all $\mathcal{A}(1)$-modules: the module $L = \mathbf{F}_2[x, x^{-1}]$ has trivial Q_0 and Q_1 homology, yet is not stably free. It provides another idempotent as well.

Proposition 11.1. *As $\mathcal{A}(1)$-modules, $L \otimes L \;\cong\; L \oplus \bigoplus_{i,j \in \mathbf{Z}} \Sigma^{4i+2j-2}\mathcal{A}(1)$.*

Proof. It is a simple matter to verify that the elements $x^{4i-1} \otimes x^{2j-1}$ generate a free submodule, and that the submodule $\{x^i \otimes x^0 | i \in \mathbf{Z}\}$, which is isomorphic to L, is a complementary submodule. □

Therefore, we have another localization functor

$$F_\infty(M) = L \otimes M.$$

The most optimistic conjecture is that this localization is the only way in which the Q_0 and Q_1 homologies fail to capture stable equivalence. A precise form of this is the following.

Conjecture 11.2. *If $F_0(M)$, $F_1(M)$ and $F_\infty(M)$ are free, then M is free.*

In a sequel to this paper, we plan to analyze this and related issues in the category $\mathcal{A}(1)$-Mod of all $\mathcal{A}(1)$-modules. In this context, we will also be able to consider the behavior of duals, which are stable inverses in the classical Picard groups, but whose behavior for modules like P_i, R and L, which are not finitely generated, is more subtle. Restricting attention to bounded below modules has precluded such considerations here.

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